Modern approaches to quantum gravity

Solution 10 Fall 2025

1. BTZ black hole

Lorentzian AdS_3 (with closed timelike curves) is defined as

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -\ell^2 \tag{1}$$

which has the global solution

$$\begin{pmatrix} X^0 \\ X^3 \end{pmatrix} = \ell \cosh\lambda \begin{pmatrix} \cos(t/\ell) \\ \sin(t/\ell) \end{pmatrix}, \quad \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \ell \sinh\lambda \begin{pmatrix} \cos(t/\ell) \\ \sin(t/\ell) \end{pmatrix}, \tag{2}$$

with $\lambda \in \mathbb{R}$, $\varphi \sim \varphi + 2\pi$, and $t \sim t + 2\pi\ell$ (this represents the closed timelike curve). The metric reads

$$ds^2 = -\cosh^2(\lambda)\, dt^2 + \ell^2\, d\lambda^2 + \ell^2 \sinh^2(\lambda)\, d\varphi^2 \tag{3}$$

and at this point we can "unroll" the time coordinate t, taking $t \in \mathbb{R}$. After passing to a new coordinate $r \in \mathbb{R}$ via

$$\sinh\lambda = \frac{r}{\ell}, \tag{4}$$

the projected metric reads

$$ds^2 = -\tilde{f}(r)\, dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2\, d\varphi^2, \quad \tilde{f}(r) = 1 + \frac{r^2}{\ell^2}. \tag{5}$$

(a) The metric of the exercise is very similar to (5), except we have

$$f_{\mathrm{BTZ}}(r) = f(-8GM; r), \quad f(c; r) := c + \frac{r^2}{\ell^2} \tag{6}$$

instead of $\tilde{f}(r)$. Notice that GM is dimensionless, since by its definition the mass dimension of G is $[G] = 2 - d = -1$ in $2+1$ spacetime dimensions. In fact, we can study the curvature tensors of the metric (5) keeping c arbitrary. Then for any c, we have the identity[1]

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{7}$$

so locally, $g_{\mu\nu}$ is maximally symmetric, with scalar curvature $R = -6/\ell^2$. Since this holds for any c, it holds in particular for f_{BTZ}.

(b) Since the metric only depends on r, it's easy to see that $\xi_{(1)} = \partial_t$ and $\xi_{(2)} = \partial_\varphi$ are Killing vectors. The global AdS_3 metric, with $c = 1$, has 6 Killing vectors, since that is the dimension of SO(2,2). However, the 4 remaining ones cease to be solutions of $\mathcal{L}_\xi g_{\mu\nu} = 0$ when $c \neq 1$.[2]

[1] This is easiest to check in Mathematica.

[2] To fully explore this claim, one should actually write down the equations $\mathcal{L}_\xi g_{\mu\nu} = 0$, which is a set of 6 equations (since $g_{\mu\nu}$ has 6 independent components) for 3 functions $(\xi^t, \xi^r, \xi^\varphi)$. This is a tedious exercise.

(c) The norm of the timelike Killing vector is

$$-\xi_{(1)} \cdot \xi_{(1)} = f_{\mathrm{BTZ}}(r) = \frac{r^2 - r_h^2}{\ell^2}. \tag{8}$$

This is positive (resp. negative) if $r > r_h$ (resp. $r < r_h$), so $r = r_h$ is indeed a Killing horizon.

(d) Let's use the Brown-York formalism. Given a Killing vector ξ^a (pushed to the boundary), the corresponding conserved charge is

$$Q[\xi] = \lim_{r\to\infty} \int_{\partial\Sigma} d^{d-2}x\sqrt{\sigma}\, n^a T_{ab}\xi^b = \lim_{r\to\infty} \frac{1}{8\pi G}\int_0^{2\pi} d\varphi \sqrt{\gamma}\, \Delta K_{ab} n^a \xi^b \tag{9}$$

with

$$\Delta K_{ab} := K_{ab} - \gamma_{ab}K - \frac{1}{\ell}\gamma_{ab}. \tag{10}$$

Here K_{ab} is the 2×2 projection of the exterior curvature $K_{\mu\nu} = -\nabla_\nu(\sigma\nu_\mu)$, the vector field $\sigma^\mu\partial_\mu = \sqrt{f_{\mathrm{BTZ}}(r)}\partial_r$ is the unit-normalized outward facing normal, and γ_{ab} is the 2×2 metric of the boundary of AdS. Finally n^a is the projection of n^μ, the unit normal orthogonal to the timeslice Σ.

A straightforward computation (keeping c arbitrary for now) gives

$$K_{\mu\nu}(r) = r\sqrt{f(r)} \times \mathrm{diag}\left(\frac{1}{\ell^2}, 0, -1\right). \tag{11}$$

This is indeed of rank 2, since $K^{\mu\nu} = 0$. Projecting to the boundary coordinates $x^a = \{t, \varphi\}$ gives the non-degenerate 2×2 matrix

$$\Delta K_{ab} = -\frac{c}{2} \times \mathrm{diag}\left(\frac{1}{\ell}, \ell\right) + O(1/r^2) \tag{12}$$

with individual components

$$\Delta K_{tt} \approx \frac{4GM}{\ell}, \quad \Delta K_{\varphi\varphi} \approx 4GM\ell, \quad \Delta K_{t\varphi} = 0. \tag{13}$$

Setting c to its BTZ value $c = -8GM$, and discarding $O(1/r^2)$ terms. In this formula, the subtraction term $(1/\ell)\times\gamma_{ab}$ is crucial — without it, terms proportional to r appear that diverge near the boundary of AdS.

Let's inspect the large-r behavior of the BY integrand (38) in more detail. The (d-2)-dimensional boundary metric is $\sigma_{\varphi\varphi} = r^2$, so $\sqrt{\sigma} = r$. Yet $n^\mu = \frac{1}{\sqrt{f_{\mathrm{BTZ}}(r)}}(1,0,0)$, so after projecting to the boundary, we have $n^a \sim (\ell/r, 0)$ as $r \to \infty$. It follows that $\sqrt{\sigma}n^a \sim (\ell, 0)$ as $r\to\infty$. Thus, if we specialize to the timelike Killing vector $\xi^a = (1,0)$, we get

$$E = \lim_{r\to\infty}\frac{1}{8\pi G}\int_0^{2\pi} d\varphi\, \sqrt{\sigma}n^t \Delta K_{tt} + O(1/r) = \frac{1}{8\pi G}\cdot 2\pi\ell\cdot\frac{4GM}{\ell} \tag{14}$$

$$= M. \tag{41}$$

Note that this is a dimensionful result; we could have defined the conserved charge with an additional factor of ℓ to make it dimensionless.

In passing, you can easily show that the spin J of the BTZ black hole from the exercise vanishes, since $\Delta K_{t\varphi} = 0$.

(e) Intrinsically, the surface gravity κ is defined as

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu \tag{42}$$

which is to be evaluated at the Killing horizon. This is awkward to compute in the metric (5), since $g_{rr} \to \infty$ precisely at $r = r_h$. Either we can change coordinates to something akin to Kruskal coordinates, or we use the formula

$$\kappa^2 = -\frac{1}{2} \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \bigg|_{r=r_h}, \tag{43}$$

which works in any coordinate system (for a derivation, see Wald, Eq. (12.5.14)).

What remains is to compute $\nabla_\mu \xi_\nu$ for the Killing vector $\xi^\mu \partial_\mu = \partial_t$. This is straightforward, for example in Mathematica. We find that

$$\nabla_r \xi_t = -\frac{r}{\ell^2}, \quad \nabla_t \xi_r = \frac{r}{\ell^2}, \tag{44}$$

and $\nabla_\mu \xi_j = 0$ for all other components. (In particular, $\nabla_\mu \xi^\mu + \nabla_\nu \xi^\mu = 0$, as befits a Killing vector.) Moreover, $g^{tt} g_{rr} = -1$, so $\nabla^\mu \xi^\nu = -\nabla_\mu \xi_\nu$ numerically. It follows that

$$\kappa^2 = -\frac{1}{2}(-1) \cdot 2 \left[\frac{r_h}{\ell^2}\right]^2 \quad \Rightarrow \quad \kappa = \frac{r_h}{\ell^2} = \sqrt{\frac{8GM}{\ell}}. \tag{45}$$

(f) The "area" A of a black hole in $2+1$d is its circumference.[3] For the BTZ black hole, this is $A = 2\pi r_h$. Hence the Bekenstein-Hawking entropy reads

$$S_{\mathrm{BH}} = \frac{A}{4G} = \frac{\pi r_h}{2G} = \sqrt{\frac{2M}{G}} \pi \ell. \tag{46}$$

(g) The Euclidean version of the BTZ geometry is

$$ds^2 = f(r) dr^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2. \tag{47}$$

where $\tau \sim \tau + \beta$ for some inverse temperature $\beta > 0$. Just as before, you can check that locally, this is a maximally symmetric space with $R = -6/\ell^2$ for any c, and in particular for $c = -8GM$. Now, the function $f_{\mathrm{BTZ}}(r)$ has a root at $r = r_h$. By the conical defect trick, we must set

$$\beta = \frac{4\pi}{f'(r_h)} \quad \Rightarrow \quad T = \frac{1}{\beta} = \frac{\sqrt{2GM}}{\pi \ell}.$$

This computes the temperature of the BTZ black hole. Finally, since $T = \kappa/(2\pi)$, we predict that

$$\kappa = 2\pi T = \frac{\sqrt{8GM}}{\ell}.$$

This matches the previous result.

[3] A black hole in a spacetime of total dimension $D = d+1$ has a surface area $A \sim R^{d-1}$ for some length scale R. At the same time, in d dimensions, the Newton constant has mass dimension $[G] = 2 - D = 1 - d$. Therefore $S = A/(4G)$ is dimensionless in any dimension, as befits an entropy.

(h) The central charge in AdS_3 is related to the AdS radius via

$$c = \frac{3\ell}{2G}.$$

We can now apply the Cardy formula with $E = M\ell$, which gives

$$S_{\mathrm{Cardy}} = \sqrt{\frac{2M}{G}\pi\ell}.$$

This matches precisely with the Bekenstein-Hawking entropy S_{BH} that you computed previously.

2. The Cardy formula

(a) We interpret $Z(\beta, R)$ as the partition function on a torus which has circumferences β and $2\pi R$. Of course we can swap the circumferences $\beta \leftrightarrow 2\pi R$ (corresponding to swapping one space with one (Euclidean) time direction) as pointed out in the main text. So that we have a torus with circumferences $2\pi R$ and β. (and now β is interpreted as the circumference of the circle on which the QFT lives, with radius $\beta/(2\pi)$, where as $2\pi R$ is the length of the compactified "time" direction.).

In a CFT, we also have scale invariance. Thus, we can rescale

$$(2\pi R, \beta) \to 2\pi\frac{R}{\beta} \cdot (2\pi R, \beta) = \left(\frac{4\pi^2 R^2}{\beta}, 2\pi R\right) \tag{15}$$

so we have a torus with circumferences $4\pi^2 R^2 \beta$ and $2\pi R$. This is interpreted as a CFT again on a spatial circle of radius R but now the time direction is compactified on a circumference $\sim 1/\beta$,

$$Z(\beta, R) = Z\left(\frac{4\pi^2 R^2}{\beta}, R\right) \tag{16}$$

This is interesting because it relates the infinite temperature limit to the low temperature limit.

(b) The easiest limit is the low temperature limit $\beta \to \infty$ where $Z = \sum_n e^{-\beta E_n}$ is dominated by the state n which lowest energy. For a unitary 2d CFT, this state has the Casimir energy $E_0 = -\frac{c}{12R}$. Thus,

$$Z(\beta, R) \approx e^{\frac{\beta c}{12R}} \qquad (\beta \to \infty) \tag{17}$$

Using (16), this gives the infinite temperature $\beta \to 0$ behaviour as well

$$Z(\beta, R) \approx e^{\frac{4\pi^2 R}{\beta}\frac{c}{12}} \qquad (\beta \to 0) \tag{18}$$

To obtain the entropy, we compute the energy density of states defined through

$$Z(\beta) = \int \rho(E) e^{-\beta E} dE \tag{19}$$

Writing $\rho(E) = e^{S(E)}$ where S is the entropy, we want to solve S such that it gives the behaviour (18). It is useful to rescale $E = \hat{E}/\beta$ so that

$$Z(\beta) = \frac{1}{\beta} \int e^{S(\hat{E}/\beta) - \hat{E}} d\hat{E} \tag{20}$$

We now perform a saddle point approximation in the large $1/\beta$ limit. The integral will be dominated around the saddle for which

$$\frac{d}{d\hat{E}} (S(\hat{E}/\beta) - \hat{E}) = 0 \tag{21}$$

It is not hard to convince oneself that $S(E) \sim \sqrt{E}$ is the correct choice to reproduce (18). Indeed, for such a choice the saddle is at $\hat{E} \sim 1/\beta$ so that it contributes $\sim e^{1/\beta}$. Fixing the factors, we obtain

$$S(E) = 2\pi \sqrt{\frac{cER}{3}} \tag{22}$$

Another quick way is to use

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z \approx \frac{\pi^2 Rc}{3\beta^2} \tag{23}$$

with the inverse relation

$$\beta = \sqrt{\frac{\pi^2 Rc}{3\langle E \rangle}} \tag{24}$$

and thus

$$S = \beta \langle E \rangle + \log Z = \frac{2\pi^2 Rc}{3\beta} = 2\pi \sqrt{\frac{cER}{3}} \tag{25}$$

3. Asymptotic charges in gauge theory

(a) The infinitesimal variation of the fields under this transformation is:

$$\delta\phi = i\epsilon\phi, \quad \delta\phi^\star = -i\epsilon\phi^\star.$$

The conserved current is:

$$j^\mu_\phi = i \left(\partial^\mu \phi^\star \phi - \phi^\star \partial^\mu \phi \right).$$

The conserved charge associated with this current on a spacelike hypersurface Σ is:

$$Q_1 = \int_\Sigma d^{D-1}x \, j_0 = \int_\Sigma d^{D-1}x \, i \left(\partial_0 \phi^\star \phi - \phi^\star \partial_0 \phi \right).$$

(b) Every step is given in the exercise statement. For more details look at the lecture notes.

(c) We find

$$M^\mu = 0 \qquad \Theta^\mu = (D^\mu\phi^*)\delta\phi + (D^\mu\phi)\delta\phi^* - F^{\mu\nu}\delta A_\nu \tag{26}$$

Therefore, using the local transformations we obtain

$$J^\mu = ig\epsilon(\phi D^\mu\phi^* - \phi^* D^\mu\phi) - F^{\mu\nu}\partial_\nu\epsilon \tag{27}$$

therefore

$$J^\mu = -\partial_\nu(F^{\mu\nu}\epsilon) + \epsilon(\partial_\nu F^{\mu\nu} + j^\mu) \tag{28}$$

where

$$j^\mu = ig\,(\phi D^\mu\phi^\star - \phi^\star D^\mu\phi)\,.$$

Using the equation of motion, this current is a total derivative on-shell.

(e) The charge can be found by integrating the current

$$Q_2[\epsilon] \;\equiv\; \int_\Sigma d\Sigma_\mu\, J^\mu$$

Using that the on-shell current is a total derivative we obtain

$$Q_2[\epsilon] \;\approx\; -\int_\Sigma d\Sigma_\mu\,\partial_\nu(\epsilon F^{\mu\nu}) \;=\; -\oint_{\partial\Sigma} d\Sigma_{\mu\nu}\,\epsilon\,F^{\mu\nu}$$

In particular, if Σ is a constant-time slice in flat space, this becomes the familiar electric-flux form

$$Q_2[\epsilon] \;\approx\; -\oint_{\Sigma'} \epsilon\,F^{0i}\,dS_i \;=\; -\oint_{\Sigma'} \epsilon\,E^i\,dS_i$$

(up to a minus sign).

(f) Charges in gauge theory are 0 if the $U(1)$ transformation is a gauge transformation, i.e. if $\epsilon(x)|_{\partial\Sigma} = 0$. On the other hand, genuine physical symmetries act non-trivially on the boundary and are characterised by $\epsilon(x)|_{\partial\Sigma} \neq 0$, and thus can have a non-zero charge.

(g) If $g \to 0$, then the fields ϕ and A^μ decouple, hence the problem is now characterised by two charges: one associated to the $U(1)$ asymptotic symmetries (also known as "large gauge transformations") of the Maxwell field, that are computed as boundary integrals, and one associated to the $U(1)$ global symmetry of the ϕ, that is no longer a sub-class of the gauge transformations.